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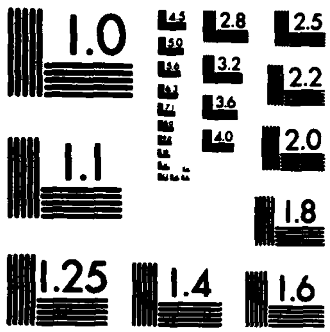

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ANALYSIS OF CONSTANT FALSE ALARM RATE SIDELOBE CANCELLER CRITERION

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I. INTRODUCTION

In this final report, the constant false alarm rate (CFAR) detection criterion for a sidelobe canceller (SLC) system, introduced in the last quarterly progress report [1], is found completely and analyzed. This new detection test for radar exhibits the desirable CFAR property that its probability of a false alarm (PFA) is functionally independent of the covariance of the actual noise field encountered. As a consequence, such a CFAR SLC system is ideally suited to cope with the newly evolving "smart" jammer threat to radar.

An important objective, set in the last quarterly progress report [1], was to find both the false alarm and signal detection probabilities of this test. The first and most important of these two goals has been met. The probability of a false alarm (or PFA) of this CFAR SLC detection criterion is derived in closed form in this report. The success in finding the PFA is due primarily to the use of a generalization of Cochran's theorem [2, p. 117].

II. FORMULATION OF THE PROBLEM

The main beam of the radar is assumed to receive a complex base-band process $y(t)$, which is sampled to yield $y(n)$ for $(n = 1, 2, \ldots, N)$. Associated with the main beam of the radar are k auxiliary antennas $x_j(t)$, called AUXs, for $(j = 1, 2, \ldots, k)$. Each AUX $x_j(t)$ is assumed to have a gain substantially less than the main beam gain--so much so, that it can be assumed that the AUXs contain no radar echo information of significance. Each AUX x_j is sampled in synchronism with the sampling of the main beam in such a manner

that the output in the j-th AUX is the sampled data set $x_j(n)$ for $(n = 1, 2, \ldots, N)$.

To restate, let $\{y(n) | n = 1, 2, \ldots, N\}$ denote a set of quadrature demodulated and sampled main beam data in which one wants to locate a desired radar signal. Also let

$$\left\{ \underline{x}(n) = \left[x_1(n), x_2(n), \ldots, x_k(n)\right]^T \middle| n = 1, 2, \ldots, N \right\}$$

denote a sampled vector set of the k AUXs of complex base-band data which is assumed to have no signals. Next, let $\{s(n), n = 1, 2, \ldots, N\}$ be the, possibly, phase and amplitude coded and sampled waveform which one would expect to find in the N samples of the main beam data. Finally, let $b = A_e^{i\delta}$ be a possible complex signal envelope, where A is the envelope and δ is the phase of signal s(n).

The radar receiver must distinguish between two hypotheses, the noise-only hypothesis, denoted by H_0, and the signal-plus-noise hypothesis, denoted by H_1. If $N_0(n)$ denotes the noise-only, sampled-data, main-beam process, and $\underline{N}(n) = \left[N_1(n), \ldots, N_k(n)\right]^T$ denotes the vector of AUX data, then

$$H_0: \begin{cases} y(n) = N_0(n) \\ \underline{x}(n) = \underline{N}(n) \text{ for } (n = 1, 2, \ldots, N), \text{ and} \end{cases}$$

$$H_1: \begin{cases} y(n) = N_0(n) + b\, s(n) \\ \underline{x}(n) = \underline{N}(n) \text{ for } (n = 1, 2, \ldots, N) \end{cases} \tag{1}$$

are the outputs of the main beam and AUX receivers under the two hypotheses.

The external noise field is primarily due to time-varying jamming, not backscatter from clutter. Thus, it can be assumed that the noise process $\left[y(t), x_1(t), \ldots, x_k(t)\right]^T$ impinging on the radar and its AUXs is a zero-mean,

Gaussian, vector process with a possibly time-varying covariance matrix, but independent from time sample to sample. For the validity of the latter assumption, time sampling needs to be accomplished at the Nyquist rate. The CFAR SLC detector criterion developed in the next section is based on the above, rather general, noise model.

III. GENERALIZED MAXIMUM LIKELIHOOD RATIO (MLR) TEST

The CFAR sidelobe canceller test of the following section uses for its derivation the generalized maximum likelihood ratio (MLR) test originally due to Neyman and Pearson. The MLR principle is best described by probabilities on a sample space X with a parameter set Ω. Let $x \in X$ be a point in sample space X. Also, let $P(x;\theta)$ be a probability density function of x, where $\theta \in \Omega$ is a parameter vector in Ω, the set of all possible values of θ.

Now suppose $w < \Omega$ is a subset and $\Omega - w$ is the subset of Ω complementary to w. Next, define two alternative hypotheses, H_0 and H_1, in set Ω in the following manner:

$$H_0 \equiv \left\{\theta \in w\right\}$$

$$H_1 \equiv \left\{\theta \in \Omega - w\right\}. \tag{2}$$

The purpose of the likelihood ratio test is to test hypothesis H_1 against H_0, holding the probability that H_1 is accepted when H_0 is actually true, i.e., the probability of a false alarm (PFA) below some fixed error level.

An important test that accomplishes the above requirement is the maximum likelihood ratio (MLR) test. See an early paper by Kelly, Reed and Root [3] for more details. Let K be some fixed threshold and define the likelihood ratio test to be

$$L(x) = \frac{\max\limits_{\theta \in \Omega - w} P(x;\theta)}{\max\limits_{\theta \in w} P(x;\theta)} \begin{array}{l} > K \text{ accept } H_1 \\ < K \text{ accept } H_0 \ , \end{array} \tag{3}$$

where hypotheses H_0 and H_1 are defined in Eq. (2).

If $\theta_i(x)$ are the maximum likelihood estimates (MLEs) of θ for a given x with respect to hypotheses $H_i\left(i = 0, 1\right)$, then L(x) in Eq. (3) is equivalent to

$$L(x) = \frac{P\left(x;\ \theta_1(x)\right)}{P\left(x;\ \theta_0(x)\right)} \ . \tag{4}$$

In this form, the likelihood ratio L(x) is explicitly related to the MLEs of the parameter vector θ under the two possible hypotheses, assuming, of course, that such estimates exist. The MLR test is used next to find an adaptive detection algorithm with the CFAR property.

Consider the (k + 1) component complex vector of main-beam and AUX data,

$$\underline{v}(n) = \begin{bmatrix} y(n) \\ \underline{x}(n) \end{bmatrix} = \left[y(n),\ x_1(n),\ \ldots,\ x_k(n)\right]^T \ , \tag{5}$$

as defined in Eq. (1) under hypothesis H_0. Then, under hypothesis H_0, the vector $\underline{v}(n)$ in Eq. (5) has the joint probability density function

$$P_0\left(\underline{v}(n)\right) = P\left(\underline{v}(n) \middle| H_0\right) = \frac{1}{(\pi)^k \lVert M \rVert} e^{-\underline{v}^*(n)\, M^{-1}\, \underline{v}(n)} \tag{6}$$

for (n = 1, 2, ..., N), where $M = E\left[\underline{v}(n)\ \underline{v}^*(n) \middle| H_0\right]$ is the unknown covariance matrix of $\underline{v}(n)$ and $\lVert M \rVert$ is its determinant. Here, the asterisk, "*", denotes conjugate matrix transpose.

The covariance matrix M in Eq. (6) was allowed to be "slowly time-varying" as a function of sampling time n. Here, M is allowed to vary in time only with

a time constant which somewhat exceeds N, the observation time in samples. Hence, under this assumption, M in Eq. (6) is approximately a constant.

Now, make the above reasonable assumption, that M is a constant over the observation time, and also the assumption made in the previous section, that the time samples are independent from sample to sample. Then, the probability density function for the N samples under hypothesis H_0 is

$$P_0(V) = P_0\left(\underline{v}(1), \underline{v}(2), \ldots, \underline{v}(N)\right)$$

$$= \prod_{n=1}^{N} P_0\left(\underline{v}(n)\right)$$

$$= \frac{1}{(\pi)^{kN} \|M\|^N} e^{-\sum_{n=1}^{N} \underline{v}^*(n) M^{-1} \underline{v}(n)} \tag{7}$$

The exponent in Eq. (7) is expressible in terms of the trace function

$$\mathrm{Tr}(A) = \sum_{i=1}^{k} a_{ii}$$

of a $k \times k$ matrix, $A = \left[a_{ij}\right]$. Since this exponent is a scaler, and since $\mathrm{Tr}(A\ B) = \mathrm{Tr}(B\ A)$, where A and B are matrices,

$$\sum_{n=1}^{N} \underline{v}^*(n) M^{-1} \underline{v}(n) = \mathrm{Tr}\left(\sum_{n=1}^{N} \underline{v}^*(n) M^{-1} \underline{v}(n)\right)$$

$$= \mathrm{Tr}\left(M^{-1} \sum_{n=1}^{N} \underline{v}(n)\ \underline{v}^*(n)\right). \tag{8}$$

Hence, in terms of Eq. (8), Eq. (7) becomes

$$P_0(V) = \frac{1}{(\pi)^{kN}\|M\|^N} e^{-\mathrm{Tr}\left(M^{-1} \sum_{n=1}^{N} \underline{v}(n)\, \underline{v}^*(n)\right)}$$

$$= \frac{1}{\pi^{kN}\|M\|^N} e^{-N\,\mathrm{Tr}\left(M^{-1}\hat{M}_0\right)} , \tag{9}$$

where

$$\hat{M}_0 = \frac{1}{N} \sum_{n=1}^{N} \underline{v}(n)\, \underline{v}^*(n) \equiv \overline{\underline{v}\, \underline{v}^*} \tag{10}$$

is the well known MLE of the unknown covariance matrix under hypothesis H_0. Here, the bar operation over the last expression for $\hat{M}_0$ denotes the time average over n. Note that the inverse of $\hat{M}_0$ exists only if $K < N$.

By Eq. (1), for the signal-plus-noise hypothesis H_1,

$$E\left\{y(n)\middle| H_1\right\} = E\left\{N_0(n)\right\} + E\left\{b\ s(n)\right\} = b\ s(n) .$$

Thus, for hypothesis H_1, the joint probability density is found in a similar fashion to be

$$P_1(V) = P_1\left(\underline{v}_b(1), \underline{v}_b(2), \ldots, \underline{v}_b(N)\right)$$

$$= \prod_{n=1}^{N} P\left(\underline{v}_b(n)\right)$$

$$= \frac{1}{\pi^{kN}\|M\|^N} e^{-N\,\mathrm{Tr}\left(M^{-1}\hat{M}_b\right)} , \tag{11}$$

where, by Eq. (1),

$$\underline{v}_b(n) = \left[y(n) - b\ s(n),\ x_1(n),\ \ldots,\ x_k(n)\right]^T , \tag{12}$$

and

$$\hat{M}_b = \frac{1}{N} \sum_{n=1}^{N} \underline{v}_b(n)\, \underline{v}_b^*(n) = \overline{\underline{v}_b\, \underline{v}_b^*} \tag{13}$$

is the MLE of the unknown covariance matrix M under hypothesis H_1.

The parameter vector θ for the MLR test in Eq. (3) consists of b, the complex signal envelope, and the $(k+2)(k+1)/2$ functionally independent elements of the unknown positive definite covariance matrix M. It is more convenient to express θ as the pair $\theta = [b, M]$ so that the sample space is

$$\Omega = \left\{\theta\right\} = \left\{[b, M] \middle| M > 0\right\}, \tag{14}$$

where $M > 0$ denotes the fact that M is Hermitian symmetric and positive definite. With this definition of sample space, the sets w and $\Omega - w$ associated with H_0 and H_1 in Eq. (2) are, respectively,

$$\begin{aligned} w &= \left\{[0, M] \middle| M > 0\right\} \\ \Omega - w &= \left\{[b, M] \middle| M > 0,\ b \neq 0\right\}. \end{aligned} \tag{15}$$

By Eqs. (10), (14), and (15), the MLEs of θ are

$$\begin{aligned} \theta_0 &= [0, \hat{M}_0] \quad \text{and} \\ \theta_1 &= [b, \hat{M}_b]\ . \end{aligned} \tag{16}$$

with respect to hypotheses H_0 and H_1, respectively, holding b fixed. A substitution of Eq. (16) into Eqs. (3) and (4) yields the MLR test

$$L(v) = \frac{\max\limits_{b} 1\Big/\|\hat{M}_b\|^N}{1\Big/\|\hat{M}_0\|^N} = \frac{\|\hat{M}_0\|^N}{\min\limits_{b}\|\hat{M}_b\|^N} \begin{array}{l} \geq K \text{ then } H_1 \\ < K \text{ then } H_0\ . \end{array}$$

This test is evidently equivalent to

$$\ell(v) = \frac{\|\hat{M}_0\|}{\min_b \|\hat{M}_b\|} \begin{array}{l} > C \text{ then } H_1 \\ < C \text{ then } H_0 \, , \end{array} \tag{17}$$

where $C = K^{1/N}$ and $\hat{M}_0$ and $\hat{M}_b$ are the matrices in Eqs. (10) and (13). The test in Eq. (17) is obtained by the same means that E. J. Kelly [4] used recently to find adaptive CFAR tests for a radar signal which is assumed to occur in only one range interval. In the present adaptive CFAR test, the radar signal is assumed to extend over many, in fact $N >> 1$, range intervals. This new test and the methods used to analyze it seem to differ somewhat from the Kelly tests.

The test in Eq. (17) is simplified by getting rid of its apparent dependence on two determinants of covariance matrices. This is to be accomplished by the well-known Shur identity (e.g., see [5, pp. 45-47]) for factoring determinants of partitioned matrices as follows:

$$\left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \left\| \begin{matrix} A - B D^{-1} C & 0 \\ C & D \end{matrix} \right\|$$

$$= \left\| A - B D^{-1} C \right\| \left\| D \right\|$$

where the square submatrix D is assumed to be nonsingular. Thus, if $N > k$, by Eqs. (12) and (13) and the Shur identity,

$$\|\hat{M}_b\| = \left\| \overline{\underline{v}_b \underline{v}_b^*} \right\| = \left\| \overline{\begin{bmatrix} y - b\,s \\ \underline{x} \end{bmatrix} \left[(y - b\,s)^* \, , \, \underline{x}^* \right]} \right\|$$

$$= \left\| \begin{matrix} \overline{|y - b\,s|^2} \, , & \overline{(y - b\,s)\,\underline{x}^*} \\ \overline{\underline{x}\,(y - b\,s)^*} \, , & \overline{\underline{x}\,\underline{x}^*} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} \overline{|y - b\,s|^2} - \overline{(y - b\,s)\,\underline{x}^*} \left(\overline{\underline{x}\,\underline{x}^*} \right)^{-1} \overline{\underline{x}\,(y - b\,s)^*} \, , & 0 \\ \overline{\underline{x}\,(y - b\,s)^*} \, , & \overline{\underline{x}\,\underline{x}^*} \end{matrix} \right\|$$

Hence,

$$\|\hat{M}_b\| = \left(\overline{|y - b\,s|^2} - \overline{(y - b\,s)\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,(y - b\,s)^*}\right)\|\hat{M}_0\|, \tag{18a}$$

where it is convenient to let $\hat{M} = \overline{x\,x^*}$. Similarly, by Eqs. (5) and (10),

$$\|\hat{M}_0\| = \left(\overline{|y|^2} - \overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right)\|\hat{M}\|. \tag{18b}$$

Substituting Eqs. (18a) and (18b) into the ratio test of Eq. (17) yields

$$\ell(v) \le \frac{\overline{|y|^2} - \overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}}{\underset{b}{\mathrm{Min}}\left(\overline{|y - b\,s|^2} - \overline{(y - b\,s)\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,(y - b\,s)^*}\right)} \begin{array}{l} \ge C \text{ for } H_1 \\ < C \text{ for } H_0 . \end{array} \tag{19}$$

An expansion of the denominator in Eq. (19) produces

$$\begin{aligned} D = {} & \overline{|y|^2} - b^*\,\overline{s\,y^*} - b\,\overline{s^*\,y} + b\,b^*\,\overline{|s|^2} \\ & - \overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*} + b\,\overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*} \\ & + b^*\,\overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*} - b\,b^*\,\overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*} \\ = {} & C - b\,B - b^*\,B^* + b\,b^*\,A , \end{aligned}$$

where

$$A = \overline{|s|^2} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*}$$

$$B = \overline{s\,y^*} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*} \text{ , and}$$

$$C = \overline{|y|^2} - \overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*} \; .$$

A completion of the square in the above quadratic form in b and b^* yields

$$\begin{aligned} D &= C + \left(\sqrt{A}\,b - B^*/\sqrt{A}\right)\left(\sqrt{A}\,b^* - B/\sqrt{A}\right) - |B|^2/A \\ &= C + \left|\sqrt{A}\,b - B^*/\sqrt{A}\right|^2 - \frac{|B|^2}{A} > 0 . \end{aligned}$$

Evidently D is minimized when

$$\sqrt{A}\, b - B^* / \sqrt{A} = 0 \quad \text{or} \quad \hat{b} = B^* / A$$

so that

$$D_{min} = C - |B|^2 / A$$

$$= \overline{|y|^2} - \overline{y\,\underline{x}^*}\;\hat{M}\;\overline{\underline{x}\,y^*} - \frac{\left|\overline{s\,y^*} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right|^2}{\overline{|s|^2} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*}} \; . \tag{20}$$

Substituting Eq. (20) into the denominator of Eq. (19) gives

$$\ell(\underline{v}) = \frac{1}{1 - \dfrac{\left|\overline{s\,y^*} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right|^2}{\left(\overline{|s|^2} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*}\right)\left(\overline{|y|^2} - \overline{y\,x^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right)}}$$

$$= \frac{1}{1 - r(\underline{v})} \geq C \quad \text{for } H_1$$

$$< C \quad \text{for } H_0 \; . \tag{21}$$

Clearly this test, Eq. (21), is equivalent to the ratio test,

$$r(\underline{v}) = \frac{\left|\overline{s\,y^*} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right|^2}{\left(\overline{|s|^2} - \overline{s\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,s^*}\right)\left(\overline{|y|^2} - \overline{y\,\underline{x}^*}\;\hat{M}^{-1}\;\overline{\underline{x}\,y^*}\right)} \geq r_o \quad \text{for } H_1$$

$$< r_o \quad \text{for } H_0 \; , \tag{22}$$

where $r_o = 1 - 1/C$.

It is now shown that the range of ratio $r(\underline{v})$ in Eq. (22) is restricted to the interval between zero and one, i.e. $0 \leq r(v) \leq 1$. In order to show this,

it is convenient to let

$$\underline{e} = \hat{M}^{-1/2}\,\underline{x}\ , \tag{23}$$

where $\hat{M}^{-1/2}$ is the inverse of the square root of $\hat{M}$. For any given $\underline{x}$ or $\hat{M} = \overline{\underline{x}\,\underline{x}^*}$, the square root may be defined by $\hat{M}^{1/2} = U\,\Lambda^{1/2}\,U^*$, where $\Lambda^{1/2}$ is diagonal with elements equal to the square roots of the eigenvalues of $\hat{M}$ and U a unitary matrix. With these definitions, one has the identity

$$\begin{aligned} \overline{\underline{e}\,\underline{e}^*} &= \hat{M}^{-1/2}\ \overline{\underline{x}\,\underline{x}^*}\ \hat{M}^{-1/2} \\ &= \hat{M}^{-1/2}\ \hat{M}\ \hat{M}^{-1/2} = I_k\ , \end{aligned} \tag{24}$$

where I_k denotes the k dimensional identity matrix, and the upper bar operation again denotes the average over the N samples.

The key identity needed to find the range of r(v) is the identity

$$\overline{s\,y^*} - \overline{s\,\underline{e}^*}\ \overline{\underline{e}\,y^*} = \overline{\left[s - \left(\overline{s\,\underline{e}^*}\right)\underline{e}\right]\left[y - \left(\overline{y\,\underline{e}^*}\right)\underline{e}\right]^*}\ . \tag{25a}$$

To prove Eq. (25a), use Eq. (24) as follows:

$$\begin{aligned} \text{right side} &= \overline{s\,y^*} - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) \\ &\quad - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{y\,\underline{e}^*}\right)^* + \left(\overline{s\,\underline{e}^*}\right)\ \overline{\underline{e}\,\underline{e}^*}\ \left(\overline{y\,\underline{e}^*}\right)^* \\ &= \overline{s\,y^*} - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) \\ &\quad + \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) \\ &= \overline{s\,y^*} - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) = \text{left side}\ . \end{aligned}$$

If one lets $y = s$ and $s = y$ in Eq. (25a), the following corollary identities obtain:

$$\overline{|s|^2} - \left(\overline{s\,\underline{e}^*}\right)\left(\overline{\underline{e}\,s^*}\right) = \overline{\left|s - \left(\overline{s\,\underline{e}^*}\right)\underline{e}\right|^2} \quad \text{and}$$

$$\overline{|y|^2} - \left(\overline{y\,\underline{e}^*}\right)\left(\overline{\underline{e}\,y^*}\right) = \overline{\left|y - \left(\overline{y\,\underline{e}^*}\right)\underline{e}\right|^2} \ . \tag{25b}$$

That the range of $r(\underline{v})$ is $0 \le r(\underline{v}) \le 1$ follows from Eqs. (22), (25a), (25b), and an evident application of Schwarz inequality to the right side of Eq. (25a), as follows:

$$\overline{\left[s - \left(\overline{s\,\underline{e}^*}\right)\underline{e}\right]\left[y - \left(\overline{y\,\underline{e}^*}\right)\underline{e}\right]}$$
$$\le \overline{\left|s - \left(\overline{s\,\underline{e}^*}\right)\underline{e}\right|^2}\ \overline{\left|y - \left(\overline{y\,\underline{e}^*}\right)\underline{e}\right|^2} \ .$$

IV. FALSE ALARM PROBABILITY FOR CFAR SLC TEST

In order to evaluate performance of the CFAR SLC test derived in the preceding section, it is important to estimate the probability of a false alarm as a function of detection threshold r_0, i.e., one needs

$$\text{PFA} = \text{Prob}\left\{r(\underline{v}) > r_0 | H_0\right\}.$$

Hence, the probability density function (PDF) of the test function $r(\underline{v})$ must be obtained in order to compute the PFA.

To derive the PDF of $r(\underline{v})$, it is convenient for preciseness to perform the averaging operations which appear in $r(\underline{v})$ as vector operations. This is accomplished by changing the functions $s = s(n)$, $y = y(n)$, $\underline{x} = \underline{x}(n)$, and $\underline{e} = \underline{e}(n)$ for $1 \le n \le N$ into vector and matrix notation, as follows:

$$\underline{s} = \left[s(1),\ s(2),\ \ldots,\ s(N)\right] \quad \text{, a row n-vector}$$

$$\underline{y} = \left[y(1),\ y(2),\ \ldots,\ y(N)\right] \quad \text{, a row n-vector}$$

$$X = \left[\underline{x}(1) \ \vdots\ \underline{x}(2) \ \vdots\ \cdots \ \vdots\ \underline{x}(N)\right] \text{, a } k \times N \text{ matrix}$$

$$J = \left[\underline{e}(1) \ \vdots\ \underline{e}(2) \ \vdots\ \cdots \ \vdots\ \underline{e}(N)\right] \text{, a } k \times N \text{ matrix .} \tag{26}$$

In terms of this new notation,

$$X\, X^* = N\, \hat{M} \quad \text{and} \quad J = \left(X\, X^*\right)^{-1/2} X \ .$$

Note in this new notation that the identity in Eq. (24) corresponds to

$$J\, J^* = \left(X\, X^*\right)^{-1/2} X\, X^* \left(X\, X^*\right)^{-1/2} = I_k \ , \tag{27a}$$

and that r in Eq. (22) is expressible as

$$r(v) = \frac{\left|\underline{s}\,\underline{y}^* - \underline{s}\, X^* \left(X\, X^*\right)^{-1} X\, \underline{y}^*\right|^2}{\left(\underline{s}\,\underline{s}^* - \underline{s}\, X^* \left(X\, X^*\right)^{-1} X\, \underline{s}^*\right)\left(\underline{y}\,\underline{y}^* - \underline{y}\, X^* \left(X\, X^*\right)^{-1} X\, \underline{y}^*\right)}$$

$$= \frac{\left|\underline{s}\,\underline{y}^* - \underline{s}\, J^*\, J\, \underline{y}^*\right|^2}{\left(\underline{s}\,\underline{s}^* - \underline{s}\, J^*\, J\, \underline{s}^*\right)\left(\underline{y}\,\underline{y}^* - \underline{y}\, J^*\, J\, \underline{y}^*\right)} \ . \tag{27b}$$

Next, note the identity

$$\underline{y}\,\underline{y}^* = \underline{y}\left(J^*\, J\right)\underline{y}^* + \left(\underline{y} - \left(\underline{y}\, J^*\right) J\right)\left(\underline{y} - \left(\underline{y}\, J^*\right) J\right)^*$$

$$= Q_1\left(\underline{y}\right) + Q_2\left(\underline{y}\right), \tag{28}$$

where Q_1 and Q_2 are quadratic forms in the components of $\underline{y}$ (see Appendix A). This identity is the same as the identity in Eq. (25b), but in the notation of Eq. (26). It follows, by expanding the last term of Eq. (28) and using Eq. (27a).

From (26) and (27), the rows of J are linearly independent--in fact, orthonormal in an inner product sense. Also, by Eq. (26), the Hermitian symmetric matrix $J^* J$ can be expressed as

$$J^* J = \begin{bmatrix} \underline{e}^*(1) \\ \underline{e}^*(2) \\ \vdots \\ \underline{e}^*(N) \end{bmatrix} \cdot J = \begin{bmatrix} \underline{e}^*(1)\, J \\ \underline{e}^*(2)\, J \\ \vdots \\ \underline{e}^*(N)\, J \end{bmatrix} ,$$

so that the i-th row of $J^* J$ is $\underline{e}^*(i)\, J$. Hence, every row of $J^* J$ is a linear combination of the k rows of J. Since the k rows of J are linearly independent, the rank of $J^* J$ is k.

For any given X or J, a unitary transformation can be found which removes the direct functional dependence of the ratio r in Eq. (27b) upon either of these matrices. The existence of such a unitary transformation is guaranteed by the generalization of Cochran's theorem given in Appendix A.

Consider any $N \times N$ unitary matrix U of form

$$U = \begin{bmatrix} \text{-----}\; J \;\text{-----} \\ u_{k+1,1}, \;\ldots,\; u_{k+1,N} \\ \vdots \\ u_{N,1}, \;\ldots,\; u_{N,N} \end{bmatrix} \tag{29a}$$

and the unitary transformation

$$\underline{g} = \underline{y}\, U^* . \tag{29b}$$

This transformation first sends the left side of Eq. (28) into

$$\underline{y}\,\underline{y}^* = \underline{g}\,U\,U^*\,\underline{g}^* = \underline{g}\,\underline{g}^* = \sum_{i=1}^{N} |g_j|^2 .$$

It next transforms the first quadratic form on the right side into

$$Q_1(y) = (\underline{y}\,J^*)(\underline{y}\,J^*) = [g_1, \ldots, g_k][g_1, \ldots, g_k]^*$$

$$= \sum_{i=1}^{k} |g_j|^2 .$$

Thus, by the same transformation, the second quadratic form is transformed into

$$Q_2(y) = \sum_{j=1}^{N} |g_j|^2 - \sum_{j=1}^{k} |g_j|^2 = \sum_{j=k+1}^{N} |g_j|^2 ,$$

so that the rank of $Q_2(\underline{y})$ is N - k, i.e., it has (N - k) non-zero eigenvalues and k eigenvalues equal to 0. Since the requirements of Cochran's theorem are satisfied, a unitary matrix of the form of Eq. (29) exists in actuality, which will transform the second quadratic form into a sum of squares in such a manner that Q_1 and Q_2 have no variable g_i in common.

Now, apply this unitary transformation to the ratio r in Eq. (27b) and, as well, transform the signal $\underline{s}$ by letting

$$\underline{t} = \underline{s}\,U^* .$$

Then, the ratio in Eq. (27b) evidently becomes

$$r = \frac{\left|\underline{t}\, U\, U^*\, \underline{g}^* - \sum_{j=1}^{k} t_j\ g_j^{\,*}\right|^2}{\left(\underline{t}\, U\, U^*\, \underline{t}^* - \sum_{i=1}^{k} \left|t_j\right|^2\right)\left(\underline{g}\, U\, U^*\, \underline{g}^* - \sum_{j=1}^{k} \left|g_i\right|^2\right)}$$

$$= \frac{\left|\sum_{j=k+1}^{N} t_j\ g_j^{\,*}\right|^2}{\left(\sum_{j=k+1}^{N} \left|t_j\right|^2\right)\left(\sum_{j=k+1}^{N} \left|g_j\right|^2\right)} \, . \tag{30}$$

For any particular J, as defined in Eq. (26), or set $\left\{\underline{x}(n)\middle| n = 1, 2, \ldots, N\right\}$ of AUX data, the unitary matrix U in Eq. (29a) and finally the vector $\underline{g}$ from $\underline{y}$ in Eq. (29b) can be determined. Thus, for the noise-only hypothesis H_0 and for a given J or X, the resulting components of $\underline{g}$ are mutually statistically independent. This follows from the assumption made in Section II, that the samples of y(n) were zero-mean Gaussian and mutually independent, and from the following argument:

$$\begin{aligned} E\left\{\underline{g}^*\ \underline{g}\middle| X, H_0\right\} &= E\left\{U\ \underline{y}^*\ \underline{y}\ U^*\middle| X, H_0\right\} \\ &= U\ E\left\{\underline{y}^*\ \underline{y}\middle| H_0\right\}\ U^* = U\left(2\sigma^2\ I_N\right)U^* \\ &= 2\sigma^2\ I_N \ , \end{aligned}$$

where σ^2 is the noise power per sampled, quadrature demodulated, component of y(n) for $1 \le n \le N$.

Now, in Eq. (30), let

$$w_j = t_j \Bigg/ \sqrt{\sum_{i=k+1}^{N} \left|t_i\right|^2}$$

for $k+1 \leq j \leq N$, then Eq. (30) simplifies further to

$$r = \frac{\left| \sum_{j=k+1}^{N} w_j \, g_j^* \right|^2}{\sum_{j=k+1}^{N} \left| g_j \right|^2} . \tag{31}$$

Another identity similar to Eq. (28) can now be used to find another unitary transformation to eliminate the functional dependence of r on the w_j in Eq. (31). This identity is

$$\sum_{i=k+1}^{N} \left| g_i \right|^2 = \left| \sum_{i=k+1}^{N} w_i \, g_i^* \right|^2 + \sum_{i=k+1}^{N} \left| g_i - \left(\sum_{i=k+1}^{N} w_i \, g_i^* \right)^* w_i \right|^2$$

$$= Q_3\left(g_{k+1}, \ldots, g_N\right) + Q_4\left(g_{k+1}, \ldots, g_N\right) . \tag{32}$$

The two quadratic forms Q_3 and Q_4 of this expression can be treated in the same manner as Q_1 in Eq. (28) to show that Q_3 has rank 1.

Next, consider any $(N - k) \times (N - k)$ unitary matrix of form

$$D = \begin{bmatrix} w_{k+1} & , w_{k+2} & , \ldots, & w_N \\ d_{k+2,\, k+1}, & d_{k+2,\, k+2}, & \ldots, & d_{k+2,\, N} \\ \vdots & \vdots & & \vdots \\ d_{N,\, k+1} & , d_{N,\, k+2} & , \ldots, & d_{N,\, N} \end{bmatrix} \tag{33a}$$

and the unitary transformation

$$\left[z_{k+1}, \ldots, z_N\right] = \left[g_{k+1}, \ldots, g_N\right] D^* . \tag{33b}$$

This transforms the left side of Eq. (32) into

$$\sum_{j=k+1}^{N} |z_j|^2$$

and Q_3 into $|z_{k+1}|^2$. Hence, Q_4 is transformed into

$$\sum_{j=k+2}^{N} |z_j|^2$$

and Q_4 has rank N - k - 1. Again, the requirements of Cochran's theorem are satisfied and a unitary matrix of form D exists which will transform both Q_3 and Q_4 into a sum of squares with no variables in common.

If one applies the unitary transformation of Eq. (33b) to ratio r in Eq. (31), one obtains

$$r(\underline{v}) = \frac{|z_{k+1}|^2}{\sum_{j=k+1}^{N} |z_j|^2} \tag{34a}$$

as the test function on the condition that X or $\underline{x}(n)$ for (n = 1, 2, ..., k) is given. By the same procedure as before, the z_j's are mutually independent. But, also by Eq. (1), under hypothesis H_0, the quadrature components of y(n) and each component of $\underline{x}(n)$ are independent. Furthermore, a unitary transformation on a complex unitary space is equivalent to an orthogonal transformation of twice the dimension. Thus, the numerator of Eq. (34a) consists of two squares of independent real variables and the denominator consists of 2 (n - k) squares of mutually independent real variables. Hence, the ratio in Eq. (33a) has the form

$$r(v) = \frac{{R_{k+1}}^2 + {I_{k+1}}^2}{\sum_{j=k+1}^{N} \left({R_j}^2 + {I_j}^2\right)}, \tag{34b}$$

where

$$z_j = R_j + i\ I_j \text{ for } (j = k+1,\ \ldots,\ n)$$

and

$$E\left[R_j^{\,2}\,\middle|\,X,\ H_0\right] = E\left[I_j^{\,2}\,\middle|\,X,\ H_0\right] = \sigma^2\ .$$

The form of the conditional random variable $r(\underline{v})$ in Eq. (33a) or (33b) suggests that the conditional probability density of $r(\underline{v})$ is a generalized Student's t or Beta density (see Ref. 2, pp. 237-243). Thus, by Ref. 2 (18.4.2),

$$\text{Prob}\left\{r < r(\underline{v}) \le r + d\,r\,\middle|\,H_0,\ X\right\} = P\left(r\,\middle|\,H_0,\ X\right) d\,r$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\,\Gamma\left(\frac{n}{2}\right)}\, r^{\frac{m}{2}-1} \left(1 - r\right)^{\frac{n}{2}-1} d\,r\ ,$$

where, for the present case, m = 2 and n = 2 (N - k) - 2. Hence,

$$P\left(r\,\middle|\,H_0,\ X\right) = \frac{\Gamma(N-k)}{\Gamma(N-k-1)}\,(1 - r)^{N-k-2}$$

$$= (N - k - 1)\,(1 - r)^{N-k-2}\ . \tag{35}$$

Since the right side of Eq. (35) is functionally independent of X or $\underline{x}(n)$ for (n = 1, 2, ..., k), the absolute probability density of $r(\underline{v})$ under hypothesis H_0 must be the same, i.e.,

$$P\left(r\,\middle|\,H_0\right) = (N - k - 1)\,(1 - r)^{N-k-2}\ . \tag{36}$$

Integrating $P\left(r\,\middle|\,H_0\right)$ from the detection threshold r_o to 1 yields

$$\begin{aligned}
PFA &= \text{Prob}\left\{r(\underline{v}) \geq r_o \middle| H_0\right\} \\
&= \int_{r_o}^{1} (N - k - 1)\,(1 - r)^{N-k-2}\,dr \\
&= (N - k - 1)\left(- \frac{(1 - r)^{N-k-1}}{N - k - 1}\Bigg|_{r_o}^{1}\right) \\
&= \left(1 - r_o\right)^{N-k-1} \qquad (37)
\end{aligned}$$

for the actual probability of a false alarm for the CFAR sidelobe canceller criterion found in Eq. (22).

V. CONCLUDING REMARKS

A general CFAR signal detection test has been found for a k - AUX sidelobe canceller in Eq. (22). For k = 0, this test reduces to the classical CFAR power ratio test (see Ref. 3, p. 326, for the first such test--the normalized periodogram test).

In Section IV, the formula for the probability of a false alarm is found and given in Eq. (37). When k = 0, this result reduces to the false alarm rate given for the usual CFAR test. The methods used to find the PFA for the test in Eq. (22) are similar to the techniques Fisher used to find the Fisher z - distrubution, e.g., see Ref. 2, sections 11.11, 29.1, and 29.2.

For a complete performance study of the CFAR SLC test developed in this report, it would be desirable to find a formula, possibly approximate, for the probability of detection. It is expected that it will involve the techniques developed and used in Refs. 6 and 4.

REFERENCES

1. Brennan, L. E., and I. S. Reed, *A Constant False Alarm Rate Sidelobe Canceller for Radar,* Third Quarterly Progress Report, submitted to the Naval Air Systems Command on Contract N00019-84-C-0074 by Adaptive Sensors, Inc., February 1985.

2. Cramer, H., *Mathematical Methods of Statistics,* Princeton University Press, 1963.

3. Kelly, E. J., I. S. Reed, and W. L. Root, "The Detection of Radar Echoes in Noise, Part I," *Society for Industrial and Applied Mathematics,* 8(2), June 1960, pp. 309-341.

4. Kelly, E. J., *An Adaptive Detection Algorithm,* a forthcoming Technical Report from MIT Lincoln Laboratory, Lexington, MA.

5. Gantmacher, F. R., *Matrix Theory, Vol. 1,* Chelsen Publishing Co., New York 1977.

6. Reed, I. S., J. D. Mallet, and L. E. Brennan, "Rapid Convergence Rate in Adaptive Arrays," *IEEE Transactions on Aerospace and Electronic Systems,* AES-10, November 1974, pp. 853-863.

Appendix A
UNITARY TRANSFORMATION

Let $x_k = \sigma_k + i\,\tau_k$ be the N components of an N dimensional unitary space with elements $\underline{x} = [x_1, x_2, \ldots, x_N]$. Consider quadratic forms in $x_1, x_2, \ldots, x_N$ of form

$$Q(x_1, x_2, \ldots, x_N) = \sum_{i,k=1}^{N} h_{ik}\, x_i\, x_k^{*} , \tag{A.1}$$

where $h_{ki} = h_{ki}^{*}$ and where the asterisk denotes complex conjugate. If $Q(x_1, \ldots, x_N) \geq 0$, Q is a <u>non-negative</u> quadratic form. If this relation equals zero only when all the x_i equal zero, Q is said to be <u>positive definite</u>. The above matrix $H = [h_{ik}]$ with the property $H = H^{*}$ is called Hermitian symmetric. If the rank r of H is less than N, H and its corresponding quadratic form Q are called <u>positive semi-definite</u>. If rank of H is r, the <u>rank</u> of its quadratic form Q is also r.

The generalization of Cochran's theorem [2, Sec. 11.11] needed to find the PDF of $r(\underline{v})$ in Eq. (27b) of Section IV is as follows:

<u>Theorem</u>: Let an identity of form

$$\sum_{i=1}^{N} |x_i|^2 = Q_1 + \ldots + Q_t$$

be given, where Q_i for $i = 1, 2, \ldots, t$ is a non-negative quadratic form in $x_1, x_2, \ldots, x_N$ of, at most, rank r_i. Then, if

$$\sum_{i=1}^{t} r_i = N ,$$

there exists a unitary matrix U and its corresponding unitary transformation $\underline{x} = \underline{y}\, U^*$, which changes each Q_i into a sum of the squares of the absolute values of the y_i, according to the relations

$$Q_1 = \sum_{1}^{r_1} |y_i|^2 \ , \ Q_2 = \sum_{r_{1+1}}^{r_1+r_2} |y_i|^2 \ , \ \ldots, \ Q_k = \sum_{N-r_t+1}^{N} |y_i|^2 \ ,$$

in such a manner that no two Q_i have a variable y_i in common and where $\underline{x} = \left[x_1, x_2, \ldots, x_N\right]$ and $\underline{y} = \left[y_1, y_2, \ldots, y_N\right]$ are row vectors and "*" denotes conjugate transpose.

This proof of this theorem is precisely the same as the proof given by Cramer [2], except that wherever a square of a quantity appears in his proof it is here a square of an absolute value. Also, replace the words "orthogonal transformation" in the Cramer proof by "unitary transformation." Every other step and statement in the proof remain the same.

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